

# Some connections between results and problems of De Giorgi, Moser and Bangert

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## Abstract

Using theorems of Bangert, we prove a rigidity result which shows how a question raised by Bangert for elliptic integrands of Moser type is connected, in the case of minimal solutions without self-intersections, to a famous conjecture of De Giorgi for phase transitions.

## 1 Introduction

The purpose of this note is to relate some problems posed by Moser [Mos86], Bangert [Ban89] and De Giorgi [DG79]. In particular, we point out that a rigidity result in a question raised by Bangert for the case of minimal solutions of elliptic integrands would imply a one-dimensional symmetry for minimal phase transitions connected to a famous conjecture of De Giorgi.

Though the proofs we present here are mainly a straightening of the existing literature, we hope that our approach may clarify some points in these important problems and provide useful connections.

### 1.1 The De Giorgi setting

A classical phase transition model (known in the literature under the names of Allen, Cahn, Ginzburg, Landau, van der Vaals, etc.) consists in the study

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of the elliptic equation

$$\Delta u = u - 3u^2 + 2u^3, \quad (1)$$

where  $u \in C^2(\mathbb{R}^n)$ . Particular solutions of (1) are the local minimizers of the associated energy functional. Namely, we define  $W(u) := u^2(1 - u)^2$  and we say that  $u \in C^2(\mathbb{R}^n, (0, 1))$  is a minimal solution of (1) if

$$\int_B |u_x|^2 + W(u) dx \leq \int_B |u_x + \varphi_x|^2 + W(u + \varphi) dx \quad (2)$$

for any  $\varphi \in C_0^\infty(B)$  and any ball  $B \subset \mathbb{R}^n$ .

Following is a celebrated question<sup>1</sup> posed in [DG79]:

**Problem [DG79]:** *Let  $u \in C^2(\mathbb{R}^n)$  be a solution of (1) in the whole  $\mathbb{R}^n$ . Suppose that  $0 < u(x) < 1$  and  $\partial_n u(x) > 0$  for any  $x \in \mathbb{R}^n$ .*

*Is it true that all the level sets of  $u$  are hyperplanes, at least if  $n \leq 8$ ?*

To the best of our knowledge, this problem is still open in its generality, and a complete answer is known only if  $n = 2$  [BCN97, GG98] and  $n = 3$  [AC00]. In these cases, indeed, the answer to the above question is positive in a much more general setting [AAC01] – in particular, no structural assumptions are needed for the nonlinearity on the right hand side of (1). When  $4 \leq n \leq 8$ , the conjecture has been proven [Sav03] under the additional assumptions that

$$\lim_{x_n \rightarrow -\infty} u(x', x_n) = 0 \quad \text{and} \quad \lim_{x_n \rightarrow +\infty} u(x', x_n) = 1.$$

If the above limits are uniform, the conjecture holds in any dimension  $n$  [Far99b, BHM00, BBG00].

The problem has also been dealt with for  $p$ -Laplacian-type operators [Far99a, DG02, VSS06], in the Heisenberg group framework [BL03] and for free boundary models [Val06].

A natural question arising from [DG79] is whether analogous statements hold for minimal solutions. We state this question in the following form:

**Problem [DG79]<sub>MIN</sub>:** *Let  $u \in C^2(\mathbb{R}^n, (0, 1))$  be a minimal solution of (1). Is it true that all the level sets of  $u$  are hyperplanes, at least if  $n$  is small enough?*

The answer to the above question is known to be positive for  $n \leq 7$  [Sav03]. We will see that Problem [DG79]<sub>MIN</sub> has some relation with another one, posed by [Ban89] for minimizers without self-intersections in the periodic elliptic integrand context.

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<sup>1</sup>To make the notation of this note uniform, we allow ourselves to slightly change the notation of [DG79]: namely, what here is  $u$ , there is  $2u - 1$ , so that the range of  $u$ , which is here  $(0, 1)$ , corresponds to  $(-1, 1)$  there.

## 1.2 The Moser-Bangert setting

Given  $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ , which is  $\mathbb{Z}$ -periodic in the first  $n + 1$  variables, one studies functions  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  that minimize the integral  $\int F(x, u, u_x) dx$  with respect to compactly supported variations, that is

$$\int_B F(x, u, u_x) dx \leq \int_B F(x, u + \varphi, u_x + \varphi_x) dx, \quad (3)$$

for any  $\varphi \in C_0^\infty(B)$  and for any ball  $B \subset \mathbb{R}^n$ .

We assume  $F \in C^{2,\varepsilon}(\mathbb{R}^{2n+1})$ , with some  $\varepsilon \in (0, 1]$ . We also suppose that  $F = F(x, u, p)$  satisfies the following appropriate growth conditions (compare with [Mos86, (3.1)]):

$$\begin{aligned} \frac{1}{c} |\xi|^2 &\leq \sum_{1 \leq i, j \leq n} F_{p_i p_j}(x, u, p) \xi_i \xi_j \leq c |\xi|^2 \\ |F_{pu}| + |F_{px}| &\leq c(1 + |p|), \\ |F_{uu}| + |F_{ux}| + |F_{xx}| &\leq c(1 + |p|^2), \end{aligned} \quad (4)$$

for any  $(x, u, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$  and any  $\xi \in \mathbb{R}^n$ , for a suitable  $c \geq 1$ .

The above assumptions ensure the ellipticity of the corresponding Euler-Lagrange equation. Under these conditions, the minimizers inherit regularity from  $F$  and they are of class  $C^{2,\varepsilon}(\mathbb{R}^n)$  (see [Mos86, page 246] for further details).

If  $u : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  and  $\bar{k} = (k, k_{n+1}) \in \mathbb{Z}^{n+1}$ , we define<sup>2</sup>  $T_{\bar{k}}u : \mathbb{R}^n \rightarrow \mathbb{R}$  as

$$T_{\bar{k}}u(x) = u(x - k) + k_{n+1}.$$

Since  $F$  is  $\mathbb{Z}^{n+1}$ -periodic,  $T$  determines a  $\mathbb{Z}^{n+1}$ -action on the set of minimizers.

We will consider the partial ordering on the set of functions for which we say that  $u < v$  if and only if  $u(x) < v(x)$  for all  $x \in \mathbb{R}^n$ . We then look at minimizers *without self-intersections*, i.e. minimizers whose  $T$ -orbit is totally ordered with respect to the above partial ordering. More explicitly, we say that a minimizer  $u$  is without self-intersections if  $T_{\bar{k}}u$  is either  $>$ ,  $<$  or  $=$   $u$ . It is readily seen that a minimizer  $u$  is without self-intersections if and only if the hypersurfaces  $\text{graph}(u) \subset \mathbb{R}^{n+1}$  have no self-intersections when projected into the standard torus  $\mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$  (and this property justifies the name given to it).

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<sup>2</sup>We will often adopt the notation of writing barred vectors for elements of  $(n + 1)$ -dimensional spaces: e.g.,  $k \in \mathbb{Z}^n$  versus  $\bar{k} \in \mathbb{Z}^{n+1}$ .

One denotes the set of minimizers without self-intersections by  $\mathcal{M}$ . For every  $u \in \mathcal{M}$ , [Mos86, Theorem 2.1] shows that  $\text{graph}(u)$  lies within universally bounded Hausdorff distance from a hyperplane: more explicitly, there exists  $C \geq 0$  such that for every  $u \in \mathcal{M}$  there exists  $\rho \in \mathbb{R}^n$  with

$$|u(x) - u(0) - \rho \cdot x| \leq C \quad (5)$$

for any  $x \in \mathbb{R}^n$ .

We set  $\bar{a}_1(u) = \bar{a}_1 := (-\rho, 1)/\sqrt{|\rho|^2 + 1} \in \mathbb{R}^{n+1}$ . Geometrically,  $\bar{a}_1(u)$  is the unit normal to the above mentioned hyperplane which has positive inner product with the  $(n+1)$ st standard coordinate vector. We recall that  $\bar{a}_1(u)$  is sometimes called *rotation vector* or *average slope* of  $u$  (the names are borrowed by analogous features in dynamical systems, see, e.g., [Mat82]).

We now briefly recall some useful *invariants* introduced by [Ban89]. To this extent we remark that if  $\bar{k} \in \mathbb{Z}^{n+1}$  and  $\bar{k} \cdot \bar{a}_1$  is  $> 0$  ( $< 0$ , respectively), then  $T_{\bar{k}}u > u$  ( $< u$ , respectively). To see this, take  $\bar{k} \in \mathbb{Z}^{n+1}$  with  $\bar{k} \cdot \bar{a}_1 > 0$  and suppose, by contradiction, that  $T_{\bar{k}}u \not\geq u$ . Then, since  $u$  is non-self-intersecting,  $T_{\bar{k}}u \leq u$  and so  $u(x - \ell k) + \ell k_{n+1} \leq u(x)$ , for any  $\ell \in \mathbb{N}$ . We thus have

$$0 \leq u(\ell k) - u(0) - \ell k_{n+1} \leq C - \ell(k_{n+1} - k \cdot \rho),$$

thanks to (5). By taking  $\ell$  large, since  $k_{n+1} > k \cdot \rho$ , one reaches the contradiction that proves the above observation.

If, on the other hand,  $\bar{k} \cdot \bar{a}_1 = 0$ , it is possible that  $T_{\bar{k}}u > u$  or  $< u$  or  $= u$ . Bangert gives a complete description of such possibilities in [Ban89, (3.3)-(3.7)]. We subsume this classification as follows:

**Proposition 1.1.** *For every  $u \in \mathcal{M}$  there exists an integer  $t = t(u) \in \{1, \dots, n+1\}$  and unit vectors  $\bar{a}_1 = \bar{a}_1(u), \dots, \bar{a}_t = \bar{a}_t(u)$  such that for  $1 \leq s \leq t$  we have*

$$\bar{a}_s \in \text{span } \bar{\Gamma}_s, \quad \text{where } \bar{\Gamma}_s = \bar{\Gamma}_s(u) := \mathbb{Z}^{n+1} \cap (\text{span}\{\bar{a}_1, \dots, \bar{a}_{s-1}\})^\perp, \quad (6)$$

and the  $\bar{a}_1, \dots, \bar{a}_t$  are uniquely determined by the following properties:

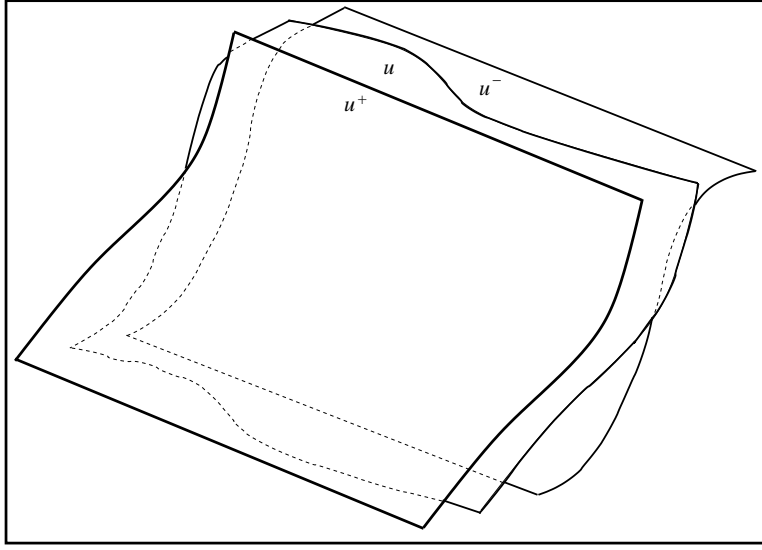
- (i)  $T_{\bar{k}}u > u$  if and only if there exists  $1 \leq s \leq t$  such that  $\bar{k} \in \bar{\Gamma}_s$  and  $\bar{k} \cdot \bar{a}_s > 0$ .
- (ii)  $T_{\bar{k}}u = u$  if and only if  $\bar{k} \in \bar{\Gamma}_{t+1}$ .

Since, as proved in [Mos86, Theorem 5.6], if  $|\bar{a}_1| = 1$  and  $\bar{a}_1 \cdot \bar{e}_{n+1} > 0$ , there always exist functions  $u \in \mathcal{M}$  with  $\bar{a}_1(u) = \bar{a}_1$ , we have that the set to which the above result applies is non-empty.

A system of unit vectors  $(\bar{a}_1, \dots, \bar{a}_t)$  is called *admissible* if  $\bar{a}_1 \cdot \bar{e}_{n+1} > 0$  and relation (6) is satisfied. For an admissible system  $(\bar{a}_1, \dots, \bar{a}_t)$  one writes

$$\mathcal{M}(\bar{a}_1, \dots, \bar{a}_t) = \{u \in \mathcal{M} \mid t(u) = t \text{ and } \bar{a}_s(u) = \bar{a}_s \text{ for } 1 \leq s \leq t\}.$$

Many results in the above setting have been obtained by [Ban89] and some of them will be needed in the sequel. For instance, in the following Proposition 1.2, we recall that for a given solution  $u$ , there exist “enveloping” solutions  $u^-$  and  $u^+$  of higher periodicity:



Enveloping solutions  $u^\pm$ .

**Proposition 1.2.** *If  $u \in \mathcal{M}(\bar{a}_1, \dots, \bar{a}_t)$  and  $t > 1$ , then there exist functions  $u^-$  and  $u^+$  in  $\mathcal{M}(\bar{a}_1, \dots, \bar{a}_{t-1})$  with the following properties:*

- (a) *If  $\bar{k}_i \in \bar{\Gamma}_t$  and  $\lim_{i \rightarrow \infty} \bar{k}_i \cdot \bar{a}_t = \pm\infty$  then  $\lim_{i \rightarrow \infty} T_{\bar{k}_i} u = u^\pm$ ,*
- (b)  *$u^- < u < u^+$  and  $T_{\bar{k}} u^- \geq u^+$  if  $\bar{k} \in \bar{\Gamma}_s$  and  $\bar{k} \cdot \bar{a}_s > 0$  for some  $1 \leq s < t$ .*

For the proof, see [Ban89, Proposition (4.2)]. A completely satisfactory uniqueness result in this framework is

**Theorem 1.3** ([Ban89], Theorem (6.22)). *If  $(\bar{a}_1, \dots, \bar{a}_t)$  is admissible, then the disjoint union  $\mathcal{M}(\bar{a}_1, ) \cup \mathcal{M}(\bar{a}_1, \bar{a}_2) \cup \dots \cup \mathcal{M}(\bar{a}_1, \dots, \bar{a}_t)$  is totally ordered.*

We point out that the proof of this theorem heavily rests on the following result, which may be seen as a uniqueness (or gap-like) statement for  $u^-$  and  $u^+$  in  $\mathcal{M}(\bar{a}_1, \dots, \bar{a}_{t-1})$ . The proof of this result is incomplete in [Ban89] and a completion is given in [JG].

**Theorem 1.4** ([Ban89], Theorem (6.6)). *If  $u \in \mathcal{M}(\bar{a}_1, \dots, \bar{a}_t)$  and  $t > 1$ , then there does not exist  $v \in \mathcal{M}(\bar{a}_1, \dots, \bar{a}_{t-1})$  such that  $u^- < v < u^+$ .*

Bangert posed a deep question in this framework in the very last paragraph of [Ban89]:

**Problem [Ban89]:** *Is it true that if  $u$  is a minimal solution and there exist  $C \geq 0$  and  $\rho \in \mathbb{R}^n$  with  $|u(x) - u(0) - \rho \cdot x| \leq C$  for any  $x \in \mathbb{R}^n$ , then  $u$  must be without self-intersections?*

We recall that the above question is known to have a positive answer when  $\bar{a}_1$  is rationally independent, cf. [Ban89, Theorem (8.4)]. Remarkably, the connection between Problems [DG79]<sub>MIN</sub> and [Ban89] will happen exactly when  $\bar{a}_1 = \bar{e}_{n+1} = (0, \dots, 0, 1)$ , which is rationally dependent.

The last notion we need to recall is the one of *foliation*. We say that a connected open subset  $G \subseteq \mathbb{R}^{n+1}$  is foliated by a subset  $\mathcal{N} \subseteq C^0(\mathbb{R}^n)$  if

$$\begin{aligned} \text{graph}(u) \cap \text{graph}(v) &= \emptyset \quad \text{for all } u, v \in \mathcal{N} \quad \text{and} \\ \bigcup_{u \in \mathcal{N}} \text{graph}(u) &= G. \end{aligned}$$

## 2 Our result

We are now in position to state the rigidity result which is the main purpose of this note. On the one hand, as we will see, the proof of it will be a simple application of the deep results already available in the existing literature. On the other hand, this result will bridge the problem of De Giorgi with the one of Bangert.

**Theorem 2.1.** *Let the setting of §1.2 hold and let  $t \in \mathbb{N}$ ,  $t \geq 2$ . Suppose that  $(\bar{a}_1, \dots, \bar{a}_t)$  is admissible and  $u_1, u_2 \in \mathcal{M}(\bar{a}_1, \dots, \bar{a}_{t-1})$  and  $u_1 < u_2$ . Assume that the set*

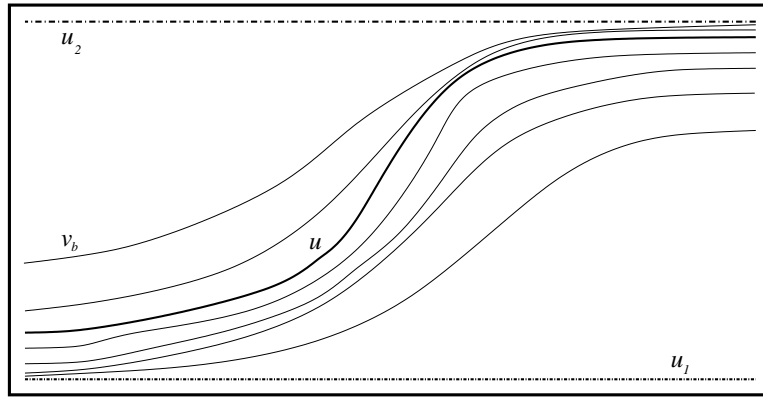
$$G = \left\{ (x, x_{n+1}) \mid u_1(x) < x_{n+1} < u_2(x) \right\} \quad (7)$$

*is foliated by a one-parameter family of functions  $(v_b)_{b \in \mathbb{R}} \subset \mathcal{M}(\bar{a}_1, \dots, \bar{a}_t)$ . Then*

$$(I) \quad (v_b)^- = u_1 \text{ and } (v_b)^+ = u_2 \text{ for every } b \in \mathbb{R}.$$

- (II) If  $u \in \mathcal{M}$  with  $u_1 < u < u_2$ , then  $t(u) \geq t$  and  $\bar{a}_i(u) = \bar{a}_i$  for any  $1 \leq i < t$ . If furthermore  $\bar{a}_t(u) = \bar{a}_t$ , then there exists  $b_0 \in \mathbb{R}$  such that  $u = v_{b_0}$ , and in particular  $t(u) = t$ .

In a verbose mode, Theorem 2.1 says the following: take an admissible set of invariants  $(\bar{a}_1, \dots, \bar{a}_t)$  and two minimal solutions  $u_1 < u_2$  without self-intersections with the above invariants except for the last one. Suppose that the space in between  $u_1$  and  $u_2$  is foliated by minimizers  $v_b$  which have all invariants  $(\bar{a}_1, \dots, \bar{a}_t)$ . Then, any minimal solution without self-intersections lying between  $u_1$  and  $u_2$  and possessing the right last invariant must agree with one of the  $v_b$ 's.



**The foliation of Theorem 2.1.**

The following results relate Problems  $[\text{DG79}]_{\text{MIN}}$  and  $[\text{Ban89}]$  in the case of minimal solutions without self-intersections of phase transition models:

**Corollary 2.2.** *Let  $u \in C^2(\mathbb{R}^n, (0, 1))$  be a minimal solution of (1). Suppose that  $T_{\bar{k}}u$  is either  $>$ ,  $<$  or  $= u$  for any  $\bar{k} \in \mathbb{Z}^{n+1}$ . Then all the level sets of  $u$  are hyperplanes.*

We denote by  $[r]$  the integer part of  $r \in \mathbb{R}$ . We then extend the potential of (2) into a (reasonably smooth) periodic one, in order to connect the setting in §1.1 with the one in §1.2.

**Corollary 2.3.** *If Problem  $[\text{Ban89}]$  has a positive answer in dimension  $n$  for  $F = |p|^2 + W([u])$  and  $\bar{a}_1 = \bar{e}_{n+1}$ , then Problem  $[\text{DG79}]_{\text{MIN}}$  has a positive answer in dimension  $n$ .*

We point out that while the setting in §1.2 only has discrete translational invariance, the one in §1.1 possesses full translational and rotational invariance: thus, in concrete cases, other phase transition models may be reduced to the setting in §1.2 after appropriate rotations and scalings.

In the De Giorgi framework, the analysis of the profile at infinity has often played a central rôle (see, e.g., [AC00, Sav03, VSS06]). Theorem 2.1 also gives some information about such asymptotic profile, according to the following result:

**Corollary 2.4.** *Suppose that  $u_1, u_2 \in \mathcal{M}(\bar{a}_1)$ ,  $u_1 < u_2$ , that for  $\bar{\Gamma}_2 = \mathbb{Z}^{n+1} \cap \text{span}\{\bar{a}_1\}^\perp$  we have  $\dim(\text{span}\bar{\Gamma}_2) = n$  and that, for any admissible pair  $(\bar{a}_1, \bar{a}_2)$ ,*

$$G = \left\{ (x, x_{n+1}) \mid u_1(x) < x_{n+1} < u_2(x) \right\}$$

*is foliated by  $(v_{\bar{a}_2, b})_{b \in \mathbb{R}} \subset \mathcal{M}(\bar{a}_1, \bar{a}_2)$ . Suppose that  $u \in C^2(\mathbb{R}^n)$  (possibly with self-intersections) is minimizing in the sense of (3) and  $u_1 < u < u_2$ . Then there exist  $w \in \mathcal{M}$  and a sequence  $\bar{k}_i \in \bar{\Gamma}_2$  such that  $T_{\bar{k}_i} u \rightarrow w$  in  $C_{\text{loc}}^1$ . Furthermore we have either  $w \equiv u_1$  or  $w \equiv u_2$  or  $w \equiv v_{\bar{a}_2, b}$  for some  $\bar{a}_2$  with  $(\bar{a}_1, \bar{a}_2)$  admissible and  $b \in \mathbb{R}$ .*

Notice that Corollary 2.4 implies some kind of limit property for minimal solutions of (1) (with possible self-intersections), in the sense that it is always possible to find directions in which  $u$  suitably approaches a “pure phase” 0 or 1:

**Corollary 2.5.** *Let  $u \in C^2(\mathbb{R}^n, (0, 1))$  be a minimal solution of (1), possibly with self-intersections. Then there exist a sequence  $\bar{k}_i \in \mathbb{R}^n \times \{0\}$  and  $\omega \in S^{n-1}$  in such a way that*

$$\lim_{t \rightarrow +\infty} \lim_{i \rightarrow +\infty} u(\omega t - k_i) \in \{0, 1\}.$$

### 3 Proofs

*Proof of Theorem 2.1.* (I) Let  $b \in \mathbb{R}$  be arbitrary. Without loss of generality we argue only for  $(v_b)^+ =: w$ . By Proposition 1.2 we have  $w \in \mathcal{M}(\bar{a}_1, \dots, \bar{a}_{t-1})$ . Since the translations  $T_{\bar{k}}$  are order preserving and  $u_1 < v_b < u_2$ , we see that  $u_1 \leq w \leq u_2$ . Since furthermore  $v_b < w$ , we obtain  $u_1 < w$ .

We want to show that  $w = u_2$ . Suppose, by contradiction, that  $w \neq u_2$ . Then Theorem 1.3 implies  $w < u_2$  and thus  $u_1 < w < u_2$ . By the assumption that  $G$  is foliated by  $(v_b)_{b \in \mathbb{R}}$ , there exists  $b_1 \in \mathbb{R}$  such that  $w(0) = v_{b_1}(0)$ . Since  $v_{b_1} \in \mathcal{M}(\bar{a}_1, \dots, \bar{a}_t)$  and  $w \in \mathcal{M}(\bar{a}_1, \dots, \bar{a}_{t-1})$ , this contradicts Theorem 1.3.



(II) Let  $b \in \mathbb{R}$  be arbitrary. We apply Proposition 1.2 (b) to  $u_1 = (v_b)^-, u_2 = (v_b)^+$  and use the fact that the translations  $T_{\bar{k}}$  are order preserving to infer that

$$T_{\bar{k}}u > T_{\bar{k}}u_1 \geq u_2 > u$$

whenever  $k \in \bar{\Gamma}_i$  and  $k \cdot \bar{a}_i > 0$  for  $1 \leq i < t$ . In view of Proposition 1.1 this implies  $t(u) \geq t$  and  $\bar{a}_i(u) = \bar{a}_i$  for  $1 \leq i < t$ .

Suppose now that the condition  $\bar{a}_t(u) = \bar{a}_t$  holds. The assumption that  $G$  is foliated by  $(v_b)_{b \in \mathbb{R}}$  implies that there exists  $b_0 \in \mathbb{R}$  such that  $u(0) = v_{b_0}(0)$  and Theorem 1.3 implies  $u = v_{b_0}$ .  $\square$

*Proof of Corollary 2.2.* Let  $F := |p|^2 + W(\lfloor u \rfloor)$ . It is easily seen that  $F \in C^{2,1}$ , that it is periodic in  $(x, u)$  and that it satisfies (4), thence the setting of §1.2 holds for such  $F$ .

Let  $u_1(x) := 0$  and  $u_2(x) := 1$ . Both  $u_1$  and  $u_2$  minimize the energy in (2). They are obviously without self-intersections and so they belong to  $\mathcal{M}(\bar{e}_{n+1})$ .

If  $u_0(t)$  is the solution of the ODE

$$\ddot{u}_0 = u_0 - 3u_0^2 + 2u_0^3$$

with  $u_0(0) = 1/2$ ,

$$\lim_{t \rightarrow -\infty} u_0(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} u_0(t) = 1,$$

then for every  $\omega \in S^{n-1}$ , the family  $(v_{\omega,b})_{b \in \mathbb{R}}$  with

$$v_{\omega,b}(x) = u_0(\omega \cdot x - b) \tag{8}$$

for any  $x \in \mathbb{R}^n$  is an extremal, and consequently minimal foliation of the set

$$\mathbb{R}^n \times (0, 1) = \{(x, x_{n+1}) \mid x \in \mathbb{R}^n, u_1(x) < x_{n+1} < u_2(x)\}.$$

A reference for the fact that extremal foliations are actually minimal foliations is e.g. [GH96, 6.3]. One readily checks that  $v_{\omega,b} \in \mathcal{M}(\bar{e}_{n+1}, \bar{\omega})$ , where  $\bar{\omega} = (\omega, 0) \in \mathbb{R}^n \times \{0\}$ .

Let  $u$  be as requested by Corollary 2.2. Since  $u$  is bounded,  $\bar{a}_1(u) = \bar{e}_{n+1}$  and so

$$\bar{\Gamma}_2(u) = \mathbb{Z}^{n+1} \cap (\text{span } \bar{a}_1(u))^\perp \subseteq \mathbb{R}^n \times \{0\}. \tag{9}$$

It then follows from Theorem 2.1(II) that  $t(u) \geq 2$ , and in particular  $u$  is non-constant. Furthermore, for  $\bar{\omega} = \bar{a}_2(u)$  there exists  $b_0 \in \mathbb{R}$  such that  $u = v_{\omega,b_0}$ . Thus the level sets of  $u$  are hyperplanes.  $\square$

*Proof of Corollary 2.3.* Let  $u \in C^2(\mathbb{R}^n, (0, 1))$  be a minimal solution of (1). If Problem [Ban89] has a positive answer in dimension  $n$  for  $F = |p|^2 + W(\lfloor u \rfloor)$  and  $\bar{a}_1 = \bar{e}_{n+1}$ , then  $u$  is without self-intersections.

Consequently, by Corollary 2.2, all the level sets of  $u$  are hyperplanes, giving that Problem [DG79]<sub>MIN</sub> has a positive answer in dimension  $n$ .  $\square$

*Proof of Corollary 2.4.* By [Mos86, Theorem 3.1, Corollary 3.2], we have that  $\sup |u_x| < \infty$  and so, by [Ban89, Theorem (8.1)], there exists a sequence  $\bar{k}_i \in \bar{\Gamma}_2$  such that  $T_{\bar{k}_i} u \rightarrow w \in \mathcal{M}$  in  $C_{\text{loc}}^1$ . Clearly,  $u_1 \leq w \leq u_2$ . We may suppose indeed that  $u_1 < w < u_2$ , otherwise we get one of the first two alternatives in the statement of Corollary 2.4. By Theorem 2.1 (II) we have  $\bar{a}_1(w) = \bar{a}_1$  and  $t(w) \geq 2$ . Furthermore together with the assumptions of Corollary 2.4 Theorem 2.1 (II) implies that  $t(w) = 2$  and that for some  $b \in \mathbb{R}$  we have  $w = v_{\bar{a}_2, b}$ , where  $\bar{a}_2 = \bar{a}_2(w)$ .  $\square$

*Proof of Corollary 2.5.* With  $u_1(x) := 0$ ,  $u_2(x) := 1$ ,  $v_{\omega, b}$  as in (8) and  $G = \mathbb{R}^n \times (0, 1)$  the assumptions of Corollary 2.4 are satisfied.  $\square$

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